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Free Vibration Analysis of Laminated Plates Using a Layerwise Theory

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Reddy's layerwise theory is used to perform free-vibration analysis of laminated plates. This theory is the most current and sophisticated theory in which full account is given to various three-dimensional effects. The results obtained from this theory are compared with those obtained from a full-fledged three-dimensional elasticity analysis and various equivalent single-layer theories that are available. These include the classical laminated plate theory (CLPT), the first-order shear deformation laminated plate theory (FSDPT), and the third-order shear deformation plate theory (THSDPT). The elasticity equations are solved by utilizing the state-space variables and the transfer matrix. A detailed analysis is carried out, by uncoupling the Navier equations, to study the various mode shapes and natural frequencies of a homogeneous transversely isotropic plate. Results are also obtained for symmetric and antisymmetric cross-ply laminates.

Introduction

IT is well known by now that the classical theory of plates introduced by Kirchhoff, and extended to composite laminates,[1,2] is inadequate in modeling the dynamic aspects of laminated composite plates. The Kirchhoff assumptions amount to treating plates to be infinitely rigid in the transverse direction by neglecting transverse strains. This theory underestimates deflection and overestimates natural frequencies of a plate. In addition, the classical plate theory is plagued with the inconsistency between the order of the governing equation and the number of boundary conditions (see Stoker[3]).

Numerous plate theories that include transverse shear deformations are documented in the literature. These shear deformation plate theories can be grouped as 1) the equivalent single-layer plate theories, 2) the layerwise plate theories, and 3) the individual-layer plate theories. Furthermore, depending on whether the variation of stress components or displacement components with respect to the thickness coordinate is assumed to be known at the outset, these theories are also referred to as either the stress-based theories or the displacement-based theories (see Reddy[2]). Two different approaches are generally adopted in the stress-based theories. In the first approach, which is known as Reissner's approach,[4] assumptions are made concerning the variation of in-plane components of the stress. The second approach is attributed to Ambartsumyan[1] in which the variation of the transverse stress components with respect to the plate thickness coordinate is assumed a priori.

In the equivalent single-layer theories (see, for example, Ref. 5), on reducing the three-dimensional elasticity problem to a two-dimensional problem, the laminate is characterized as an equivalent, homogeneous layer. Therefore, the number of governing equations is not dependent on the number of plies comprising a laminate. Although these theories are more accurate than the classical laminated plate theory in predicting the global behavior of a laminate, they are usually inadequate in describing the stress field in a laminate at the ply level. For a review of the various equivalent single-layer theories that have appeared in the open literature, see Reddy.[6]

On the other hand, in the layerwise theories[7] and individual-layer theories,[8] the total number of governing equations is dependent on the number of layers in a laminate. If the thickness extensibility is accounted for in such theories, these theories are sometimes referred to as two-and-half-dimensional theories.[9] The major difference between the layerwise theories and individual-layer theories lies in the fact that in the former theories the continuity of the transverse normal and shear stresses is not forced at the interfaces of any two adjacent layers. In the layerwise theory of Reddy[7] considered in this work, the transverse strains are not required to be continuous at the interfaces, which leaves the possibility that the transverse stresses are continuous at the interfaces of dissimilar material layers. This is particularly true when each physical layer is modeled with two or more mathematical layers. For this reason, computationally more involved individual-layer theories will not be considered in this work. However, the natural frequencies of laminated plates according to an individual-layer theory[8] will be compared with those obtained within various equivalent single-layer theories, a layerwise theory, and the three-dimensional elasticity theory. Throughout this work the following theories will be considered: 1) the third-order shear deformation laminated plate theory of Reddy,[5] 2) the first-order shear deformation laminated plate theory, 3) the classical laminated plate theory, and 4) the layerwise laminated plate theory of Reddy.[7]

Theoretical Formulation

Equivalent Single-Layer Laminated Plate Theories

The presentation of the equivalent single-layer theories used throughout this work can best be accomplished by considering the third-order shear deformation plate theory (TSDPT) of Reddy.[5] In Reddy's TSDPT the components of the displacement vector at a generic point in the plate are expressed in the form

$$
\begin{aligned}
u_1(x, y, z, t) &= u + z\psi_x - \lambda \frac{4}{3h^2} z^3\left(\psi_x + \frac{\partial w}{\partial x}\right) \\
u_2(x, y, z, t) &= v + z\psi_y - \lambda \frac{4}{3h^2} z^3\left(\psi_y + \frac{\partial w}{\partial y}\right) \\
u_3(x, y, z, t) &= w
\end{aligned}
\tag{1}
$$

where z denotes the thickness coordinate, and the generalized displacements u, v, ψ_x, ψ_y, and w are functions of the in-plane

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coordinates x and y and time t only. In Eqs. (1), h denotes the total thickness of the laminate. The displacement field in Eqs. (1) is chosen such that the transverse stress free boundary conditions on the bounding planes of a plate (laminated of orthotropic layers) are identically satisfied. In Eqs. (1), λ is a parameter introduced to combine the first-order shear deformation plate theory [FSDPT ($\lambda=0$)] and Reddy's TSDPT ($\lambda=1$) into one theory. That is, when $\lambda=0$ Eqs. (1) are reduced to the displacement field of FSDPT. Finally, by letting $\psi_x=-\partial w/\partial x$, $\psi_y=-\partial w/\partial y$ the displacement field of the classical plate theory (CLPT) is recovered. The equations of motion of a laminated plate based on the displacement field given in Eq. (1) derived using Hamilton's principle are given in Ref. 10 and hence are not being given here.

The equivalent single-layer theories are often adequate for predicting the global response quantities of a laminated plate and may not describe the correct stress field at the ply level. This is due to the fact that in most shear deformation equivalent single-layer theories the following assumptions are made: the plate is transversely inextensible, and the transverse stress components are discontinuous through the laminate thickness. In the next section we review Reddy's layerwise plate theory,[7] which removes the first assumption explicitly and leaves a possibility for the transverse stresses to be continuous through the thickness.

Layerwise Laminated Plate Theory of Reddy

Before we present the pertinent equations of motion of a generally laminated plate we consider a two-layered plate as shown in Fig. 1. Let $u^1(x, y, t)$, $u^2(x, y, t)$, and $u^3(x, y, t)$ represent the displacement components of all points located at $z=-h/2$, $z=0$, and $z=h/2$, respectively, in the x direction. If we further assume that the displacement component of the plate in the x direction has a linear variation within each layer, we have

$$u_1(x, y, z, t)=\begin{cases}[1+(2/h)z]u^2(x, y, t)-(2/h)zu^1(x, y, t) & 0\geq z\geq -h/2\\ (2/h)zu^3(x, y, t)+[1-(2/h)z]u^2(x, y, t) & 0\leq z\leq -h/2\end{cases}\tag{2}$$

Similar expressions can also be written for the displacement components $u_2(x, y, z, t)$ and $u_3(x, y, z, t)$ of a material point located at (x, y, z) in the undeformed laminate in the y and z directions. This way the displacement field will be continuous through the laminate thickness; the transverse strain components, however, will not be continuous at the interface of the two layers. On the other hand, this leaves the possibility of the transverse stress components becoming continuous at the interface of the two layers. This seemingly hypothetical presumption can become a reality by subdividing each physical layer into a finite number of mathematical layers, that is, by introducing more interfaces and, therefore, more unknown generalized displacement components at such mathematical interfaces.

It is possible to generalize the result in Eq. (2) for a generally laminated plate by representing $u_1(x, y, z, t)$, $u_2(x, y, z, t)$, and $u_3(x, y, z, t)$ as

$$\begin{aligned}u_1(x, y, z, t)&=u^i(x, y, t)\cdot\phi^i(z)\\ u_2(x, y, z, t)&=v^i(x, y, t)\cdot\phi^i(z)\\ u_3(x, y, z, t)&=w^i(x, y, t)\cdot\phi^i(z)\\ i&=1, 2, \ldots, N+1\end{aligned}\tag{3}$$

where $\phi^i(z)$ are defined as (see Fig. 2)

$$\phi^i(z)=\begin{cases}0 & z\leq z_{i-1}\\ \psi_1^i=(-1/h_{i-1})(z_{i-1}-z) & z_{i-1}\leq z\leq z_i\\ \psi_2^i=(-1/h_i)(z-z_{i+1}) & z_i\leq z\leq z_{i+1}\\ 0 & z\geq z_{i+1}\end{cases}\tag{4}$$

In Eqs. (3) the function $u^i(x, y, t)$, $v^i(x, y, t)$, and $w^i(x, y, t)$ represent the displacement components of all points located on the ith plane (defined by $z=z_i$, see Fig. 2) in the x, y, and z directions, respectively, in the undeformed laminate. Also N denotes the total number of mathematical layers considered in a laminate. Note that a repeated index indicates summation over all values of that index.

Substituting Eqs. (3) in three-dimensional strain-displacement relations results in

$$\begin{gathered}\epsilon_{11}=\frac{\partial u^i}{\partial x}\cdot\phi^i,\qquad \epsilon_{22}=\frac{\partial v^i}{\partial y}\cdot\phi^i,\qquad \gamma_{12}=\left(\frac{\partial u^i}{\partial y}+\frac{\partial v^i}{\partial x}\right)\cdot\phi^i\\ \gamma_{13}=u^i\cdot\frac{\mathrm{d}\phi^i}{\mathrm{d}z}+\frac{\partial w^i}{\partial x}\cdot\phi^i,\qquad \gamma_{23}=v^i\cdot\frac{\mathrm{d}\phi^i}{\mathrm{d}z}+\frac{\partial w^i}{\partial y}\phi^i\\ \epsilon_{33}=w^i\cdot\frac{\mathrm{d}\phi^i}{\mathrm{d}z}\end{gathered}\tag{5}$$

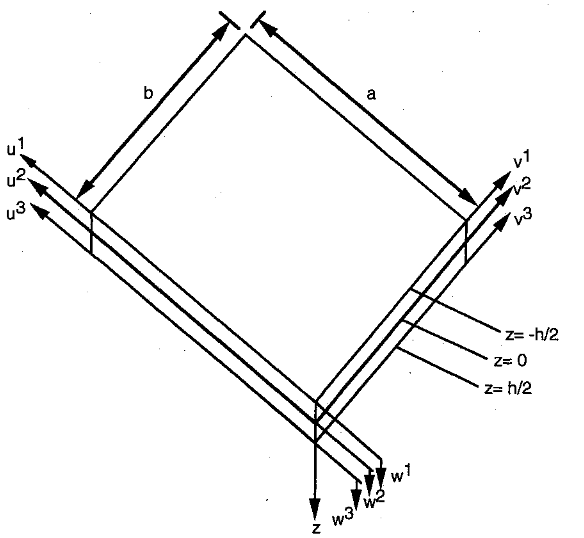

Fig. 1 Displacement components $u^j(x, y, t)$, $v^j(x, y, t)$, and $w^j(x, y, t)$ in a two-layered plate in the layerwise theory.

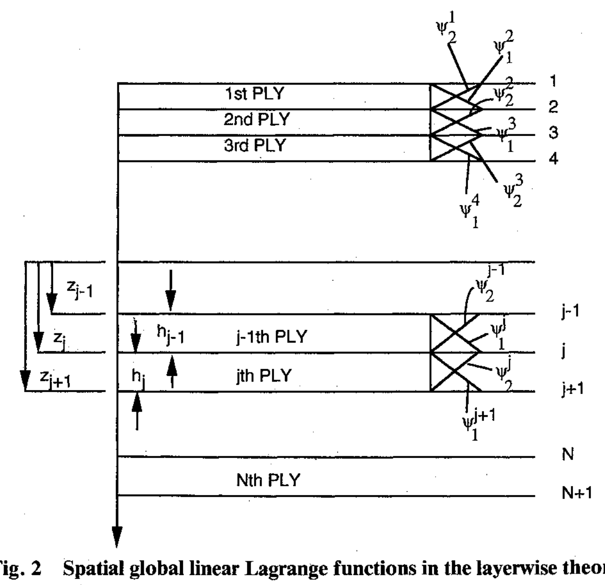

Fig. 2 Spatial global linear Lagrange functions in the layerwise theory.

Using Hamilton's principle, $3(N+1)$ equations of motion corresponding to $3(N+1)$ unknowns u^i, v^i, and w^i are derived

$$\delta u^i: \quad \frac{\partial M_1^i}{\partial x} + \frac{\partial M_6^i}{\partial y} - Q_1^i = I^{ij}\ddot{u}^j$$

$$\delta v^i: \quad \frac{\partial M_6^i}{\partial x} + \frac{\partial M_2^i}{\partial y} - Q_2^i = I^{ij}\ddot{v}^j \qquad (6)$$

$$(i, j = 1, 2, \ldots, N+1)$$

$$\delta w^i: \quad \frac{\partial K_1^i}{\partial x} + \frac{\partial K_2^i}{\partial y} - Q_3^i + \delta_{i1}P_z = I^{ij}\ddot{w}^j$$

where δ_{i1} is the Kroneckar delta. The generalized stress resultants M_1^i, M_2^i, etc., and mass terms I^{ij} are defined as

$$(M_1^i, M_2^i, M_6^i) = \int_{-h/2}^{h/2} (\sigma_{11}, \sigma_{22}, \sigma_{12}) \cdot \phi^i \, \mathrm{d}z$$

$$(Q_1^i, Q_2^i, Q_3^i) = \int_{-h/2}^{h/2} (\sigma_{13}, \sigma_{23}, \sigma_{33}) \cdot \frac{\mathrm{d}\phi^i}{\mathrm{d}z} \, \mathrm{d}z \qquad (7a)$$

$$(K_1^i, K_2^i) = \int_{-h/2}^{h/2} (\sigma_{13}, \sigma_{23}) \cdot \phi^i \, \mathrm{d}z$$

$$I^{ij} = \int_{-h/2}^{h/2} \rho \phi^i \phi^j \, \mathrm{d}z \qquad (7b)$$

where $\rho(x, y, z)$ denotes the mass density of the material point located at (x, y, z) in a laminate. For a laminated plate with a rectangular planform, the boundary conditions in layerwise plate theory (LWPT) at an edge parallel to x axis involves the specification of u^i or M_1^i, v^i or M_6^i, and w^i or K_1^i. Similarly, at an edge parallel to y axis, we can specify the required boundary conditions.

In the equivalent single-layer theories we invoked the plane-stress assumption to express the generalized stress resultants in terms of displacement components. In LWPT, however, we remove this assumption and use the three-dimensional constitutive law

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}^k = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix}^k \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}^k \qquad (8)$$

where $\bar{C}_{ij}^{(k)}$ are the transformed material stiffnesses of the kth layer. On substitution of Eqs. (5) into (8) and the subsequent results into Eqs. (7a) we obtain

$$M_1^i = A_{11}^{ij} \frac{\partial u^j}{\partial x} + A_{12}^{ij} \frac{\partial v^j}{\partial y} + B_{13}^{ji} w^j$$

$$M_2^i = A_{12}^{ij} \frac{\partial u^j}{\partial x} + A_{22}^{ij} \frac{\partial v^j}{\partial y} + B_{23}^{ji} w^j$$

$$M_6^i = A_{66}^{ij} \left(\frac{\partial u^j}{\partial y} + \frac{\partial v^j}{\partial x} \right)$$

$$Q_1^i = D_{55}^{ij} u^j + B_{55}^{ij} \frac{\partial w^j}{\partial x}$$

$$Q_2^i = D_{44}^{ij} v^j + B_{44}^{ij} \frac{\partial w^j}{\partial y} \qquad (9)$$

$$Q_3^i = B_{13}^{ij} \frac{\partial u^j}{\partial x} + B_{23}^{ij} \frac{\partial v^j}{\partial y} + D_{33}^{ij} w^j$$

$$K_1^i = B_{55}^{ji} u^j + A_{55}^{ij} \frac{\partial w^j}{\partial x}$$

$$K_2^i = B_{44}^{ji} v^j + A_{44}^{ij} \frac{\partial w^j}{\partial y}$$

where the rigidity terms A_{pq}^{ij}, B_{pq}^{ij}, and D_{pq}^{ij} are given by

$$A_{pq}^{ij} = A_{pq}^{ji} = \int_{-h/2}^{h/2} \bar{C}_{pq}^{(k)} \phi^i \phi^j \, \mathrm{d}z, \qquad p, q = 1, 2, 4, 5, 6$$

$$B_{pq}^{ij} = \int_{-h/2}^{h/2} \bar{C}_{pq}^{(k)} \frac{\mathrm{d}\phi^i}{\mathrm{d}z} \phi^j \, \mathrm{d}z \neq B_{pq}^{ji}, \qquad p, q = \overline{1, 6}$$

$$D_{pq}^{ij} = D_{pq}^{ji} = \int_{-h/2}^{h/2} \bar{C}_{pq}^{(k)} \frac{\mathrm{d}\phi^i}{\mathrm{d}z} \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \, \mathrm{d}z, \qquad p, q = 4, 5 \qquad (10)$$

By carrying out the integrations in Eqs. (10) and (7b) the rigidity and mass terms are found to be

$$A_{pq}^{ij} = \begin{cases} \tfrac{1}{3} h_i \bar{C}_{pq}^{(i)} + \tfrac{1}{3} h_{i-1} \bar{C}_{pq}^{(i-1)} & \text{if } i = j \\ \tfrac{1}{6} \bar{C}_{pq}^{(i)} \cdot h_i & \text{if } j = i+1 \\ 0 & \text{if } j > i+1 \end{cases}$$

$$B_{pq}^{ij} = \begin{cases} -\tfrac{1}{2} \bar{C}_{pq}^{(i)} + \tfrac{1}{2} \bar{C}_{pq}^{(i-1)} & \text{if } i = j \\ -\tfrac{1}{2} \bar{C}_{pq}^{(i)} & \text{if } j = i+1 \\ \tfrac{1}{2} \bar{C}_{pq}^{(i-1)} & \text{if } j = i-1 \\ 0 & \text{if } j > i+1 \text{ and } j < i-1 \end{cases}$$

$$D_{pq}^{ij} = \begin{cases} \dfrac{1}{h_i} \bar{C}_{pq}^{(i)} + \dfrac{1}{h_{i-1}} \bar{C}_{pq}^{(i-1)} & \text{if } i = j \\ \dfrac{1}{h_i} \bar{C}_{pq}^{(i)} & \text{if } j = i+1 \\ 0 & \text{if } j > i+1 \end{cases}$$

$$I^{ij} = \begin{cases} \tfrac{1}{3} h_i \rho^{(i)} + \tfrac{1}{3} h_{i-1} \rho^{(i-1)} & \text{if } i = j \\ \tfrac{1}{6} h_i \rho^{(i)} & \text{if } j = i+1 \\ 0 & \text{if } j > i+1 \end{cases} \qquad (11)$$

The free-vibration analysis of general cross-ply laminates will be considered in the next section, where the equations of motion will also be presented in terms of the generalized displacement components.

Free-Vibration Analysis: Comparison with Three-Dimensional Elasticity

The results will be developed for cross-ply laminates. For completeness, a brief account of the solution methodology of Navier's equation of motion will be given for such a class of laminates. A close theoretical examination will also be made to clarify the accuracy of the equivalent single-layer theories by comparison with elasticity equations. This comparison will be made here for a homogeneous transversely isotropic plate. The significance of LWPT will become apparent when the numerical results for natural frequencies are compared with those obtained within the three-dimensional elasticity theory.

Natural Frequencies According to Various Plate Theories

For general cross-ply laminates, the following coefficients of the layerwise plate theory are zero: $A^{ij}_{r6} = B^{ij}_{r6} = 0$, $r = 1, 3$, $A^{ij}_{45} = D^{ij}_{45} = 0$. Also, in the equivalent single-layer theories, the following laminate stiffnesses are identically zero: $A_{i6} = B_{i6} = D_{i6} = E_{i6} = F_{i6} = H_{i6} = 0$, $i = 1, 2$, and $A_{45} = D_{45} = F_{45} = 0$.

The equations of motion of a laminate according to LWPT [Eq. (6)] can be expressed in terms of the generalized displacement functions as

$$A^{ij}_{11}u^j_{,xx} + A^{ij}_{66}u^j_{,yy} - D^{ij}_{55}u^j + (A^{ij}_{12} + A^{ij}_{66})v^j_{,xy} + (B^{ji}_{13} - B^{ij}_{55})w^j_{,x} = I^{ij}\ddot{u}^j \tag{12a}$$

$$(A^{ij}_{12} + A^{ij}_{66})u^j_{,xy} + A^{ij}_{66}v^j_{,xx} + A^{ij}_{22}v^j_{,yy} - D^{ij}_{44}v^j + (B^{ji}_{23} - B^{ij}_{44})w^j_{,y} = I^{ij}\ddot{v}^j \tag{12b}$$

$$(B^{ji}_{55} - B^{ij}_{13})u^j_{,x} + (B^{ji}_{44} - B^{ij}_{23})v^j_{,y} + A^{ij}_{55}w^j_{,xx} + A^{ij}_{44}w^j_{,yy} - D^{ij}_{33}w^j + \delta_{i1}\cdot P_z = I^{ij}\ddot{w}^j \qquad i, j = 1, 2, 3, \ldots, N+1 \tag{12c}$$

where a comma followed by a variable indicates differentiation with respect to that variable. Similarly, the equations of motion of equivalent single-layer theories can be expressed in terms of the generalized displacement functions as

$$[L]\{\Delta\} = \{F\} \tag{13a}$$

where the linear operators L_{ij} for TSDPT, FSDPT, and CLPT are displayed in Appendix A of Ref. 11. The displacement vector $\{\Delta\}$ and force vector $\{F\}$ are defined as

$$\{\Delta\}^T = \{u, v, \psi_x, \psi_y, w\} \tag{13b}$$

$$\{F\}^T = \{0, 0, 0, 0, P_z\} \tag{13c}$$

for TSDPT and FSDPT, and for CLPT

$$\{\Delta\}^T = \{u, v, w\} \tag{13d}$$

$$\{F\}^T = \{0, 0, P_z\} \tag{13e}$$

With $P_z = 0$, Eqs. (12) and the appropriate boundary conditions define the differential eigenvalue problem of a plate based on LWPT. Assuming

$$\begin{Bmatrix} u^j \\ v^j \\ w^j \end{Bmatrix} = \begin{Bmatrix} U^j_{mnk} \\ V^j_{mnk} \\ W^j_{mnk} \end{Bmatrix} \cdot \cos \omega_{mnk}t \tag{14a}$$

into Eqs. (12), with $P_z = 0$, we obtain

$$A^{ij}_{11}U^j_{mnk,xx} + A^{ij}_{66}U^j_{mnk,yy} - D^{ij}_{55}U^j_{mnk} + (A^{ij}_{12} + A^{ij}_{66})V^j_{mnk,xy} + (B^{ji}_{13} - B^{ij}_{55})W^j_{mnk,x} = -\omega^2_{mnk}I^{ij}U^j_{mnk}$$

$$(A^{ij}_{12} + A^{ij}_{66})U^j_{mnk,xy} + A^{ij}_{66}V^j_{mnk,xx} + A^{ij}_{22}V^j_{mnk,yy} - D^{ij}_{44}V^j_{mnk} + (B^{ji}_{23} - B^{ij}_{44})W^j_{mnk,y} = -\omega^2_{mnk}I^{ij}V^j_{mnk}$$

$$(B^{ji}_{55} - B^{ij}_{13})U^j_{mnk,x} + (B^{ji}_{44} - B^{ij}_{23})V^j_{mnk,y} + A^{ij}_{55}W^j_{mnk,xx} + A^{ij}_{44}W^j_{mnk,yy} - D^{ij}_{33}W^j_{mnk} = -\omega^2_{mnk}I^{ij}W^j_{mnk} \tag{14b}$$

Solution of Eqs. (14b), subject to homogeneous boundary conditions, results in the natural frequencies ω_{mnk} and the eigenfunctions U^j_{mnk}, V^j_{mnk}, and W^j_{mnk}. We assume that the rectangular plate is simply supported along its edges. Hence at $x = 0$ and a we impose the following conditions.

LWPT:

$$v^j = w^j = M^j_1 = 0 \qquad j = 1, 2, \ldots, N+1 \tag{15a}$$

TSDPT:

$$N_1 = v = \psi_y = w = M_1 = P_1 = 0 \tag{15b}$$

FSDPT:

$$N_1 = v = \psi_y = w = M_1 = 0 \tag{15c}$$

CLPT:

$$N_1 = v = w = M_1 = 0 \tag{15d}$$

Similar, appropriate boundary conditions are imposed at $y = 0, b$.

In free-vibration analysis these conditions are identically satisfied if we let

$$\begin{Bmatrix} u^j \\ v^j \\ w^j \end{Bmatrix} = \begin{Bmatrix} U^j_{mnk} \\ V^j_{mnk} \\ W^j_{mnk} \end{Bmatrix} \cos \omega_{mnk}t = \begin{Bmatrix} A^j_{mnk} \cos \alpha_m x \sin \beta_n y \\ B^j_{mnk} \sin \alpha_m x \cos \beta_n y \\ C^j_{mnk} \sin \alpha_m x \sin \beta_n y \end{Bmatrix} \cos \omega_{mnk}t \tag{16}$$

in LWPT and

$$\begin{Bmatrix} u \\ v \\ \psi_x \\ \psi_y \\ w \end{Bmatrix} = \begin{Bmatrix} U_{mnk} \\ V_{mnk} \\ \Psi_{x_{mnk}} \\ \Psi_{y_{mnk}} \\ W_{mnk} \end{Bmatrix} \cos \omega_{mnk}t = \begin{Bmatrix} A_{mnk} \cos \alpha_m x \sin \beta_n y \\ B_{mnk} \sin \alpha_m x \cos \beta_n y \\ C_{mnk} \cos \alpha_m x \sin \beta_n y \\ D_{mnk} \sin \alpha_m x \cos \beta_n y \\ E_{mnk} \sin \alpha_m x \sin \beta_n y \end{Bmatrix} \cos \omega_{mnk}t \tag{17}$$

in the equivalent single-layer theories. In Eqs. (16) and (17) we have

$$\alpha_m = \frac{m\pi}{a} \qquad \text{and} \qquad \beta_n = \frac{n\pi}{b} \tag{18}$$

where m and n are the Fourier integers.

Substitution of Eqs. (16) into Eqs. (14b) results in the generalized eigenvalue system

$$[S]\{\bar{\Delta}\} = \omega^2_{mnk}[M]\{\bar{\Delta}\} \tag{19}$$

The elements of stiffness matrix $[S]$ and mass matrix $[M]$ are presented in Ref. 11 and

$$\{\bar{\Delta}\}^T = \{A^1_{mnk}, A^2_{mnk}, \ldots, A^{N+1}_{mnk}, B^1_{mnk}, B^2_{mnk}, \ldots, B^{N+1}_{mnk}, C^1_{mnk}, C^2_{mnk}, \ldots, C^{N+1}_{mnk}\} \tag{20}$$

On solving Eqs. (19) we obtain, for each pair of integers m and n, k $(= 3N + 3)$ frequencies. On the other hand, solution of Navier's equations for each pair of m and n will result in an infinite number of frequencies. It will become apparent that the natural frequencies determined within LWPT, for each pair of m and n, are the approximate values of $3N + 3$ lowest natural frequencies obtained from the exact three-dimensional elasticity theory.

Substitution of Eqs. (17) into Eqs. (13) results in a generalized eigenvalue system similar to the one in Eqs. (19) where $\{\bar{\Delta}\}^T = \{A_{mnk}, B_{mnk}, C_{mnk}, D_{mnk}, E_{mnk}\}$, for TSDPT and FSDPT, and $\{\bar{\Delta}\}^T = \{A_{mnk}, B_{mnk}, E_{mnk}\}$ for CLPT. The elements of $[S]$ and $[M]$ in TSDPT, FSDPT, and CLPT are also displayed in Ref. 11. Solution of the eigenvalue system [Eqs. (19)] for each pair of m and n will result in k $(=5)$ natural frequencies within TSDPT and FSDPT and k $(=3)$ natural frequencies within CLPT. These frequencies will also be the approximate values of the lowest natural frequencies of the exact theory.

For a symmetrically laminated plate, the linear stretching and bending equations of motion according to equivalent single-layer theories are uncoupled. This is due to the vanishing of coupling stiffnesses of the laminate. In the layerwise plate theory, however, since the stiffnesses have a local nature, i.e., they are defined in terms of material and geometric properties of adjacent layers, see Eqs. (11), this uncoupling does not occur. This is the case also for the individual-layer plate theories and three-dimensional elasticity equations. For this uncoupling to occur, further assumptions must be made concerning the variations of the displacement components with respect to thickness coordinate z.

Natural Frequencies According to Three-Dimensional Elasticity Theory

The exact free-vibration analysis of orthotropic laminates is carried out by Srinivas et al.[12] However, the numerical results are tabulated for a three-ply symmetric laminate. Here, we develop a more general procedure for solving the exact three-dimensional elasticity equations and generate extensive numerical results for both symmetric and antisymmetric cross-ply laminates. The exact results will be compared with those obtained by various plate theories. This way a more accurate assessment can be obtained concerning the validity of the plate theories for cross-ply laminates.

We let $u^k(x, y, z, t)$, $v^k(x, y, z, t)$, and $w^k(x, y, z, t)$ denote the displacement components of a material point located at (x, y, z) in the kth ply of a general cross-ply laminate (see Fig. 3) in the x, y, and z directions, respectively. By treating each ply as an individual homogeneous plate, Navier's equation of motion for the kth ply can be presented as

$$
\begin{aligned}
&\bar{C}^k_{11}u^k_{,xx} + \bar{C}^k_{66}u^k_{,yy} + \bar{C}^k_{55}u^k_{,zz} + (\bar{C}^k_{12} + \bar{C}^k_{66})v^k_{,xy} \\
&\quad + (\bar{C}^k_{13} + \bar{C}^k_{55})w^k_{,xz} = \rho^k \ddot{u}^k \\
&(\bar{C}^k_{12} + \bar{C}^k_{66})u^k_{,xy} + \bar{C}^k_{66}v^k_{,xx} + \bar{C}^k_{22}v^k_{,yy} + \bar{C}^k_{44}v^k_{,zz} \\
&\quad + (\bar{C}^k_{23} + \bar{C}^k_{44})w^k_{,yz} = \rho^k \ddot{v}^k \\
&(\bar{C}^k_{13} + \bar{C}^k_{55})u^k_{,xz} + (\bar{C}^k_{23} + \bar{C}^k_{44})v^k_{,yz} + \bar{C}^k_{55}w^k_{,xx} \\
&\quad + \bar{C}^k_{44}w^k_{,yy} + \bar{C}^k_{33}w^k_{,zz} = \rho^k \ddot{w}^k
\end{aligned}
\tag{21}
$$

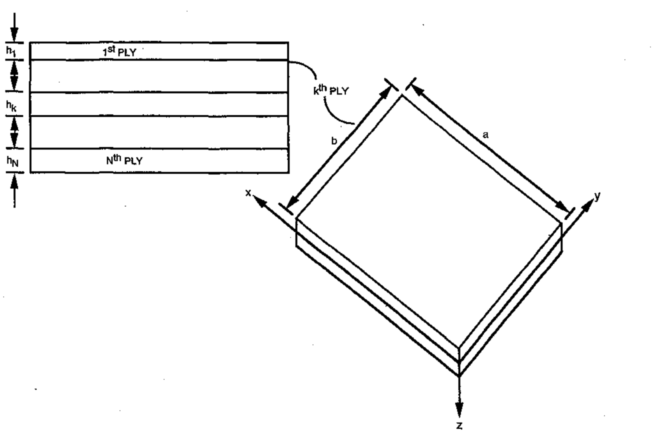

Fig. 3 Geometry and coordinates of a laminate for the exact elasticity solution.

The stress-displacement relations in the kth ply are obtained by merely substituting the linear strain-displacement relations into Eqs. (8)

$$
\begin{aligned}
\sigma^k_{11} &= \bar{C}^k_{11}u^k_{,x} + \bar{C}^k_{12}v^k_{,y} + \bar{C}^k_{13}w^k_{,z} \\
\sigma^k_{12} &= \bar{C}^k_{66}(u^k_{,y} + v^k_{,x}) \\
\sigma^k_{22} &= \bar{C}^k_{12}u^k_{,x} + \bar{C}^k_{22}v^k_{,y} + \bar{C}^k_{23}w^k_{,z} \\
\sigma^k_{23} &= \bar{C}^k_{44}(v^k_{,z} + w^k_{,y}) \\
\sigma^k_{33} &= \bar{C}^k_{13}u^k_{,x} + \bar{C}^k_{23}v^k_{,y} + \bar{C}^k_{33}w^k_{,z} \\
\sigma^k_{13} &= \bar{C}^k_{55}(u^k_{,z} + w^k_{,x})
\end{aligned}
\tag{22}
$$

At the four edges of laminate simply supported conditions are assumed. That is, at $x = 0$, a: $v = w = \sigma_{11} = 0$; and at $y = 0$, b: $u = w = \sigma_{22} = 0$ for all z. These conditions are satisfied by assuming

$$
\begin{Bmatrix} u^k(x, y, z, t) \\ v^k(x, y, z, t) \\ w^k(x, y, z, t) \end{Bmatrix} = \begin{Bmatrix} U^k_{mn}(z) \cdot \cos \alpha_m x \sin \beta_n y \\ V^k_{mn}(z) \cdot \sin \alpha_m x \cos \beta_n y \\ W^k_{mn}(z) \cdot \sin \alpha_m x \sin \beta_n y \end{Bmatrix} \cdot \cos \omega_{mn} t
\tag{23}
$$

where α_m and β_n are defined in Eq. (18). For the sake of convenience we introduce the state variables

$$
\begin{aligned}
Z^k_1(z) &= U^k_{mn}(z), \qquad Z^k_2(z) = \frac{\mathrm{d}U^k_{mn}}{\mathrm{d}z} = \frac{\mathrm{d}Z^k_1}{\mathrm{d}z} \\
Z^k_3(z) &= V^k_{mn}(z), \qquad Z^k_4(z) = \frac{\mathrm{d}V^k_{mn}}{\mathrm{d}z} = \frac{\mathrm{d}Z^k_3}{\mathrm{d}z} \\
Z^k_5(z) &= W^k_{mn}(z), \qquad Z^k_6(z) = \frac{\mathrm{d}W^k_{mn}}{\mathrm{d}z} = \frac{\mathrm{d}Z^k_5}{\mathrm{d}z}
\end{aligned}
\tag{24}
$$

Substitution of Eqs. (23) into Eqs. (21) and introducing Eqs. (24) result in a system of six coupled first-order ordinary differential equations which can be presented as

$$
\left\{\frac{\mathrm{d}Z^k}{\mathrm{d}z}\right\} = [A^k]\{Z^k\}
\tag{25}
$$

where

$$
\{Z^k\} = \begin{Bmatrix} Z^k_1 \\ Z^k_2 \\ \vdots \\ Z^k_6 \end{Bmatrix}
\tag{26}
$$

The constant elements of the coefficient matrix $[A^k]$ are displayed in Ref. 13. The general solution of Eqs. (25) is given by (e.g., see Ref. 14)

$$
\{Z^k\} = [U^k][Q^k(z)]\ \{B^k\}
\tag{27}
$$

where $B^k_1, B^k_2, \ldots, B^k_6$ are six arbitrary unknown, and generally complex, constants and

$$
[Q^k_z] = \begin{bmatrix} e^{\lambda^k_1 z} & & & & 0 \\ & e^{\lambda^k_2 z} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & e^{\lambda^k_6 z} \end{bmatrix}
\tag{28}
$$

In Eqs. (28) and (27), $\lambda^k_j\,(j = \overline{1, 6})$ and $[U^k]$ are the eigenvalues and the matrix of eigenvectors (modal matrix) of the coefficient matrix $[A^k]$. These eigenvalues and eigenvectors, in general, can be complex valued. Equations (27) can alternatively be presented as

$$Z_i^k = \sum_{j=1}^{6} u_{ij}^k e^{\lambda_j^k z} B_j^k, \qquad i = 1, 2, \ldots, 6 \tag{29}$$

To obtain the frequency equation of the laminate, we must satisfy the continuity and equilibrium conditions at the interfaces and the stress-free boundary conditions at the top and bottom surfaces of the laminate. Assuming that the z-coordinate is located at the middle surface of each ply (see Fig. 3) and that the thickness of kth ply is h_k, we have the following conditions.

Continuity conditions:

$$\begin{aligned} u^k(x, y, \bar{h}_k, t) &= u^{k+1}(x, y, -\bar{h}_{k+1}, t) \\ v^k(x, y, \bar{h}_k, t) &= v^{k+1}(x, y, -\bar{h}_{k+1}, t) \\ w^k(x, y, \bar{h}_k, t) &= w^{k+1}(x, y, -\bar{h}_{k+1}, t) \end{aligned} \tag{30a}$$

Equilibrium conditions:

$$\begin{aligned} \sigma_{13}^k(x, y, \bar{h}_k, t) &= \sigma_{13}^{k+1}(x, y, -\bar{h}_{k+1}, t) \\ \sigma_{23}^k(x, y, \bar{h}_k, t) &= \sigma_{23}^{k+1}(x, y, -\bar{h}_{k+1}, t) \\ \sigma_{33}^k(x, y, \bar{h}_k, t) &= \sigma_{33}^{k+1}(x, y, -\bar{h}_{k+1}, t) \end{aligned} \tag{30b}$$

where

$$\bar{h}_k = h_k/2, \qquad k = 1, 2, \ldots, N-1 \tag{31}$$

On substitution of Eqs. (29) into Eqs. (23) and the subsequent results into Eqs. (30), we obtain

$$[T_P^k]\{B^k\} = [T_m^{k+1}]\{B^{k+1}\} \tag{32}$$

where

$$\begin{aligned} [T_P^k] &= [\bar{T}^k][Q^k(\bar{h}_k)] \\ [T_m^k] &= [\bar{T}^k][Q^k(-\bar{h}_k)] \end{aligned} \tag{33}$$

and $[Q^k(\bar{h}_k)]$ and $[Q^k(-\bar{h}_k)]$ are simply obtained by replacing z in Eqs. (28) by $\bar{h}_k$ and $-\bar{h}_k$, respectively. The elements of $[\bar{T}^k]$ are given in Ref. 13.

From Eqs. (32) we have

$$\begin{aligned} \{B^1\} &= [T_p^1]^{-1}[T_m^2]\{B^2\} \\ \{B^2\} &= [T_p^2]^{-1}[T_m^3]\{B^3\} \\ &\vdots \\ \{B^{N-1}\} &= [T_p^{N-1}]^{-1}[T_m^N]\{B^N\} \end{aligned} \tag{34}$$

Hence, by the process of elimination, we can express $\{B^1\}$ in terms of $\{B^N\}$ as

$$\{B^1\} = [S]\{B^N\} \tag{35}$$

where the transfer matrix $[S]$ is given by

$$[S] = [T_p^1]^{-1}[T_m^2][T_p^2]^{-1}[T_m^3] \cdots [T_p^{N-1}]^{-1}[T_m^N] \tag{36}$$

It is to be noted that, from Eqs. (33), we have

$$[T_m^k][T_p^k]^{-1} = [\bar{T}^k][Q^k(-h_k)][\bar{T}^k]^{-1} \tag{37}$$

which can be used in the computation of transfer matrix $[S]$.

The frequency determinant of the laminate is finally obtained by satisfying the traction-free boundary conditions on the bounding planes of the laminate. These conditions are

$$\begin{aligned} \sigma_{13}^1(x, y, -\bar{h}_1, t) &= \sigma_{23}^1(x, y, -\bar{h}_1, t) \\ &= \sigma_{33}^1(x, y, -\bar{h}_1, t) = 0 \end{aligned} \tag{38a}$$

and

$$\begin{aligned} \sigma_{13}^N(x, y, \bar{h}_N, t) &= \sigma_{23}^1(x, y, \bar{h}_N, t) \\ &= \sigma_{33}^N(x, y, \bar{h}_N, t) = 0 \end{aligned} \tag{38b}$$

Substitution of Eqs. (29) and (23) into Eqs. (38a) and (38b) results in

$$\underbrace{[R^1]}_{3\times 6}\ \underbrace{[Q^1(-\bar{h}_1)]}_{6\times 6}\ \{B^1\} = \{0\} \tag{39a}$$

and

$$\underbrace{[R^N]}_{3\times 6}\ \underbrace{[Q^N(\bar{h}_N)]}_{6\times 6}\ \{B^N\} = \{0\} \tag{39b}$$

where the matrices $[R^1]$ and $[R^N]$ are defined in Ref. 13. Substituting Eqs. (35) into Eqs. (39a), we obtain

$$[R^1][Q^1(-\bar{h}_1)][S]\{B^N\} = \{0\} \tag{40}$$

Equations (40) and (39b) comprise a system of six simultaneous, homogeneous, algebraic equations which can be presented as

$$\begin{bmatrix} [M_T] \\ [M_B] \end{bmatrix} \{B^N\} = \{0\} \tag{41}$$

where

$$\begin{aligned} [M_T] &= [R^1][Q^1(-\bar{h}_1)][S] \\ [M_B] &= [R^N][Q^N(\bar{h}_N)] \end{aligned} \tag{42}$$

To obtain a nontrivial solution, the determinant of the coefficient matrix in Eqs. (41) must be set equal to zero. However, this determinant is complex valued. To transform it to a real-valued determinant we follow the procedure developed by Nosier and Reddy.[13] To this end, we note that from Eqs. (27) and (28) we have

$$\{B^N\} = [U^N]^{-1}\{Z^N(0)\} \tag{43}$$

Substitution of Eqs. (43) into (41) results in

$$\begin{bmatrix} [M_T] \\ [M_B] \end{bmatrix} [U^N]^{-1}\{Z^N(0)\} = \{0\} \tag{44}$$

For a nontrivial solution, we must set the determinant of the real-valued coefficient matrix in Eq. (44) equal to zero. That is

$$\left| \begin{bmatrix} [M_T] \\ [M_B] \end{bmatrix} \right| \Big/ |[U^N]| = 0 \tag{45}$$

It is to be noted that for each pair of integers m and n Eq. (45) will yield an infinite number of the laminate's natural frequencies. These frequencies correspond to different vibrational modes of the laminate. This point will be closely elaborated in the next section.

Special Study: Transversely Isotropic Plates

For a closer examination of the plate theories, it is best to consider the vibration of a homogeneous plate. To this end, we consider a plate made of transversely isotropic material; the plane of isotropy is assumed to be parallel to the x-y plane. It will be shown that for such a plate there exist two distinct classes of vibrational modes according to the exact three-dimensional theory. In the first class the transverse displacement function $u_3(x, y, z, t)$ vanishes everywhere in the plate. We will refer to these modes as the zero-w modes of the plate. The second class of modes corresponds to the vanishing of a potential function $\Phi(x, y, z, t)$ (to be defined in a later section) everywhere in the plate. These modes will be referred to as the zero-Φ modes of the plate. Furthermore, it will be shown that there exist two different kinds of zero-w modes. In the first kind, the in-plane displacement components $u_1(x, y, z, t)$ and $u_2(x, y, z, t)$ are symmetric with respect to the plate thickness coordinate z. The second kind of zero-w modes correspond to the modes for which u_1 and u_2 are antisymmetric with respect to the z coordinate. There are also two different kinds of zero-Φ modes. For the first kind, u_1 and u_2 are antisymmetric and the transverse displacement function is symmetric with respect to the thickness coordinate. In the second kind of zero-Φ modes, on the other hand, u_1 and u_2 are symmetric and the transverse displacement function is antisymmetric. Before we present the exact analysis, we first review the equations of equivalent single-layer plate theories and present the natural frequencies of the plate according to these theories.

Plate Equations

As we pointed out earlier, the equations of motion of symmetrically laminated plates in the equivalent single-layer plate theories are uncoupled into two sets of equations. The first set characterizes the stretching problem whereas the second set of equations describes the transverse bending problem of the plate. For example, assuming that the symmetric plate is a laminate consisting of transversely isotropic layers, the equations of motion according to TSDPT [Eqs. (13)] can be presented as follows.[15]

Stretching equations:

$$\hat{M}u_{,xx} + \frac{\hat{B}}{2}u_{,yy} + \left(\hat{M} - \frac{\hat{B}}{2}\right)v_{,xy} = m_1\ddot{u}$$
$$\left(\hat{M} - \frac{\hat{B}}{2}\right)u_{,xy} + \frac{\hat{B}}{2}v_{,xx} + \hat{M}v_{,yy} = m_1\ddot{v} \tag{46}$$

Bending equations:

$$D\psi_{x,xx} + C\psi_{x,yy} + (D - C)\psi_{y,xy} - A(\psi_x + w_{,x}) - F\nabla^2 w_{,x} = m_3\ddot{\psi}_x - m_5\ddot{w}_{,x} \tag{47a}$$

$$D\psi_{y,yy} + C\psi_{y,xx} + (D - C)\psi_{x,xy} - A(\psi_y + w_{,y}) - F\nabla^2 w_{,y} = m_3\ddot{\psi}_y - m_5\ddot{w}_{,y} \tag{47b}$$

$$(F\nabla^2 + A)(\psi_{x,x} + \psi_{y,y}) - H\nabla^2\nabla^2 w + A\nabla^2 w + P_z = m_1\ddot{w} + m_5(\ddot{\psi}_{x,x} + \ddot{\psi}_{y,y}) - m_7\nabla^2\ddot{w} \tag{47c}$$

where ∇^2 is the two-dimensional Laplace operator. The stiffnesses $\hat{M}, \hat{B}, \ldots, A$ and the mass terms $m_1, \ldots, m_7$ are defined in Appendix C of Ref. 11. It is to be noted that the stretching equations of the plate according to FSDPT and CLPT are identical to those in Eqs. (46) (Ref. 13).

As we noted, Eqs. (47) are three coupled partial differential equations in terms of the rotation functions ψ_x and ψ_y and the transverse displacement function w. These equations can alternatively be formulated into two uncoupled equations. This is accomplished by introducing a potential function as[15]

$$\Phi(x, y, t) = \psi_{x,y} - \psi_{y,x} \tag{48}$$

and the resulting equations are

$$(1/A)(\hat{D}H - \bar{F}^2)\nabla^2\nabla^2\nabla^2 w - \hat{D}\nabla^2\nabla^2 w + P_z - (D/A)\nabla^2 P_z + (m_3/A)\ddot{P}_z = m_1\ddot{w} - [\hat{m}_3 + (D/A)m_1]\nabla^2\ddot{w}$$
$$+ (1/A)(\hat{m}_3 H + m_7\hat{D} - 2\bar{m}_5\bar{F})\nabla^2\nabla^2\ddot{w} + (m_1 m_3/A)\ddddot{w} + (1/A)(\bar{m}_5^2 - \hat{m}_3 m_7)\nabla^2\ddddot{w} \tag{49a}$$

$$C\nabla^2\Phi - A\Phi = m_3\ddot{\Phi} \tag{49b}$$

where the additional mass and stiffness terms introduced are defined in Ref. 13. Equations (49a) and (49b) are known as the interior and edge-zone equations of a plate, respectively. The alternate formulation of bending equations of FSDPT is also presented in Ref. 13. For a simply supported plate the boundary conditions according to TSDPT are given by Eqs. (15b). The first two conditions in Eqs. (15b) imply the specification of $u_{,x} = v = 0$ at $x = 0, a$; which supplement the stretching equations (46). In free-vibration problems the last four conditions in Eqs. (15b) can be stated as[15]

$$w = w_{,xx} = w_{,xxxx} = 0; \qquad \Phi_{,x} = 0 \qquad \text{at } x = 0, a \tag{50}$$

Hence, the displacement function w and the potential function Φ are also not coupled in the boundary terms. Similar boundary conditions can also be written in terms of u, v, w, and Φ at $y = 0$ and b (Ref. 15).

Natural Frequencies According to Plate Equations

As a result of the preceding uncouplings, we conclude that the vibration modes (and frequencies) of a simply supported plate can be categorized into two distinct classes. The frequencies and modes of the first class are obtained from the stretching and the edge-zone equations and correspond to vanishing of the transverse displacement w. The frequencies of the zero-w modes are obtained by introducing

$$u(x, y, t) = A_{mn}\cos\alpha_m x \sin\beta_n y \cdot \cos\omega_{mn}t$$
$$v(x, y, t) = B_{mn}\sin\alpha_m x \cos\beta_n y \cdot \cos\omega_{mn}t \tag{51}$$

$$\Phi(x, y, t) = C_{mn}\cdot\cos\alpha_m x \cos\beta_n y \cdot \cos\omega_{mn}t \tag{52}$$

into Eqs. (46) and (49b), respectively,

$$\omega_{mn}^2 = (\hat{B}/2)m_1(\alpha_m^2 + \beta_n^2) \tag{53a}$$

$$\omega_{mn}^2 = (\hat{M}/m_1)(\alpha_m^2 + \beta_n^2) \tag{53b}$$

and

$$\omega_{mn}^2 = (1/m_3)[A + C(\alpha_m^2 + \beta_n^2)] \tag{54}$$

The frequencies in Eqs. (53) are known as the stretching (extensional) frequencies, and the frequency in Eq. (54) is referred to as the thick-twist frequency.[16] Finally, two frequencies of the zero-Φ modes are determined by solving

$$[m_1 m_3 - (\bar{m}_5^2 - \hat{m}_3 m_7)(\alpha_m^2 + \beta_n^2)](\omega_{mn}^2)^2 - [m_1 A + (\hat{m}_3 H + m_7\hat{D} - 2\bar{m}_5\bar{F})(\alpha_m^2 + \beta_n^2)^2$$
$$+ (A\hat{m}_3 + Dm_1)(\alpha_m^2 + \beta_n^2)](\omega_{mn}^2) + (\hat{D}H - \bar{F}^2)(\alpha_m^2 + \beta_n^2)^3 + A\hat{D}(\alpha_m^2 + \beta_n^2)^2 = 0 \tag{55}$$

which is obtained by introducing

$$w(x, y, t) = D_{mn}\sin\alpha_m x \sin\beta_n y \cdot \cos\omega_{mn}t \tag{56}$$

into the interior equation (49), with $P_z = 0$. These frequencies are referred to as flexural and flexural thickness-shear frequencies.[16] The stretching frequencies of the plate according to FSDPT and CLPT are identical to those given in Eqs. (52). In summary, the remaining frequencies of the plate according to these theories are obtained from the following equations (see Nosier and Reddy[17]).

Thickness-twist frequency (FSDPT):

$$\omega_{mn}^2 = (1/\hat{m}_3)[K^2\hat{A} + \hat{C}(\alpha_m^2 + \beta_n^2)] \tag{57}$$

Flexural and thickness-shear frequencies (FSDPT):

$$m_1\hat{m}_3(\omega_{mn}^2)^2 - [m_1K^2\hat{A} + (K^2\hat{A}\hat{m}_3 + \hat{D}m_1)(\alpha_n^2 + \beta_n^2)](\omega_{mn}^2) + K^2\hat{A}\hat{D}(\alpha_n^2 + \beta_n^2)^2 = 0 \tag{58}$$

Flexural frequency (CLPT):

$$\omega_{mn}^2 = \hat{D}/m_1(\alpha_m^2 + \beta_n^2)^2 \tag{59}$$

where $K^2(=K_4^2 = K_5^2)$ denotes the shear correction factor. For a single-layer transversely isotropic plate we have

$$\begin{aligned} &\hat{B} = 2Gh, \quad \hat{M} = [E/(1-\nu^2)]h, \quad A = (8/15)G_zh \\ &C = (17/325)Gh^3, \quad \hat{A} = G_zh, \quad \hat{C} = (1/12)Gh^3 \\ &m_1 = \rho h, \quad m_3 = (17/315)\rho h^3, \quad \hat{m}_3 = (1/12)\rho h^3 \end{aligned} \tag{60}$$

where E, ν, . . . , G_z are the engineering material properties defined in Ref. 15. Substitution of Eqs. (60) into Eqs. (53), (54), and (57) results in the following.

Stretching frequencies:

TSDPT, FSDPT, and CLPT

$$\frac{\rho}{G}\omega_{mn}^2 = (\alpha_m^2 + \beta_n^2) \tag{61}$$

TSDPT, FSDPT, and CLPT

$$\frac{\rho}{G}\omega_{mn}^2 = \frac{2}{1-\nu}(\alpha_m^2 + \beta_n^2) \tag{62}$$

Thickness-twist frequency:

TSDPT

$$\frac{\rho}{G}\omega_{mn}^2 = \frac{168}{17}\frac{G_z}{G}\frac{1}{h^2} + (\alpha_m^2 + \beta_n^2) \simeq 1.00129\frac{G_z}{G}\left(\frac{\pi}{h}\right)^2 + (\alpha_m^2 + \beta_n^2) \tag{63}$$

FSDPT

$$\frac{\rho}{G}\omega_{mn}^2 = K^2\left(\frac{12}{\pi^2}\right)\frac{G_z}{G}\left(\frac{\pi}{h}\right)^2 + (\alpha_m^2 + \beta_n^2) \tag{64}$$

Similarly, the flexural frequencies according to TSDPT, FSDPT, and CLPT are obtained by substituting Eq. (60) into (55), (58), and (59), respectively.

Elasticity Equations

To closely examine the results obtained in the previous section, we reconsider Navier's equations in more detail. For a homogeneous transversely isotropic medium the equations are

$$C_{11}u_{,xx} + ½(C_{11} - C_{12})u_{,yy} + C_{44}u_{,zz} + ½(C_{11} + C_{12})v_{,xy} + (C_{13} + C_{44})w_{,xz} = \rho\ddot{u} \tag{65a}$$

$$½(C_{11} + C_{12})u_{,xy} + ½(C_{11} - C_{12})v_{,xx} + C_{11}v_{,yy} + C_{44}v_{,zz} + (C_{13} + C_{44})w_{,yz} = \rho\ddot{v} \tag{65b}$$

$$(C_{13} + C_{44})(u_{,x} + v_{xy})_{,z} + C_{44}\nabla^2 w + C_{33}w_{,zz} = \rho\ddot{w} \tag{65c}$$

where ∇^2 is the two-dimensional Laplace operator.

For such a medium the stress-displacement relations are given as

$$\sigma_{13} = C_{44}(u_{,z} + w_{,x}) \tag{66a}$$

$$\sigma_{23} = C_{44}(v_{,z} + w_{,y}) \tag{66b}$$

$$\sigma_{33} = C_{13}(u_{,x} + v_{,y}) + C_{33}w_{,z} \tag{66c}$$

$$\sigma_{11} = C_{11}u_{,x} + C_{12}v_{,y} + C_{13}w_{,z} \tag{67a}$$

$$\sigma_{22} = C_{12}u_{,x} + C_{11}v_{,y} + C_{13}w_{,z} \tag{67b}$$

For a simply supported plate the boundary conditions can be stated as

$$v = w = u_{,x} = 0 \quad \text{at} \quad x = 0 \text{ and } a \tag{68a}$$

$$u = w = v_{,y} = 0 \quad \text{at} \quad y = 0 \text{ and } b \tag{68b}$$

It is to be noted that the conditions $u_{,x} = 0$ and $v_{,y} = 0$ are implied by $\sigma_{11} = 0$ and $\sigma_{22} = 0$, respectively. The traction-free conditions ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$) at the bounding planes of the plate also imply that [see Eqs. (66)]

$$u_{,z} = -w_{,x} \tag{69a}$$

$$v_{,z} = -w_{,y} \tag{69b}$$

$$(u_{,x} + v_{,y}) = -(C_{33}/C_{13})w_{,z} \quad \text{at} \quad z = \pm h/2 \tag{69c}$$

The differential eigenvalue problem described by Eqs. (65), (68), and (69) can be solved by the procedure outlined earlier. However, to better understand the results of the equivalent single-layer theories, we use an alternative approach.

Elasticity Equations: Alternate Formulation

We uncouple the Navier equations (65) by taking the partial derivatives of Eqs. (65a) and (65b) with respect to y and x, respectively, and subtract the results to obtain

$$\frac{1}{2}(C_{11} - C_{12})\nabla^2\Phi + C_{44}\frac{\partial^2\Phi}{\partial z^2} = \rho\ddot{\Phi} \tag{70}$$

where

$$\Phi(x, y, z, t) = u_{,y} - v_{,x} \tag{71}$$

Next we differentiate Eqs. (65a) and (65b) with respect to x and y, respectively, and add the results

$$\left(C_{44}\frac{\partial^2}{\partial z^2} + C_{11}\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right)(u_{,x} + v_{,y}) = -(C_{13} + C_{44})\nabla^2 w_{,z} \tag{72}$$

Now operating on Eqs. (65c) and (72) by operators $[C_{44}(\partial^2/\partial z^2) + C_{11}\nabla^2 - \rho(\partial^2/\partial t^2)]$ and $[(C_{13} + C_{44})(\partial/\partial z)]$, respectively, and subtracting the results, we obtain

$$\left(C_{44}\frac{\partial^2}{\partial z^2} + C_{11}\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right) \times \left(C_{33}\frac{\partial^2}{\partial z^2} + C_{44}\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right)w - (C_{13} + C_{44})^2\nabla^2 w_{,zz} = 0 \tag{73}$$

Hence, we replaced the three coupled Navier's equations (65) by two equivalent uncoupled equations, Eqs. (70) and (73). It remains then to express all of the boundary conditions in terms of Φ and w. To accomplish this, we first substitute Eqs. (71) into Eqs. (65a) and (65b) and obtain

$$C_{11}(u_{,x} + v_{,y})_{,x} + ½(C_{11} - C_{12})\Phi_{,y} + C_{44}u_{,zz} + (C_{13} + C_{44})w_{,xz} = \rho\ddot{u} \tag{74a}$$

$$C_{11}(u_{,x}+v_{,y})_{,y} - ½(C_{11}-C_{12})\Phi_{,x} + C_{44}u_{,zz} + (C_{13}+C_{44})w_{,xz} = \rho\ddot{v} \tag{74b}$$

Also, with the help of Eq. (65c), elimination of the term $C_{44}(\partial^2/\partial z^2)(u_{,x}+v_{,y})$ from Eq. (72) results in

$$\left(C_{11}\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right)(u_{,x}+v_{,y}) = \frac{-1}{C_{13}+C_{44}} \times [(C_{13}^2 + 2C_{13}C_{44})\nabla^2 w_{,z} + C_{44}(\rho\ddot{w}_{,z} - C_{33}w_{,zzz})] \tag{75}$$

Now evaluating Eq. (74b) at $x = 0$ (or a) and using the information in Eq. (74a) results in

$$\Phi_{,x} = 0 \qquad \text{at} \quad x = 0, a \tag{76a}$$

Similarly, evaluating Eqs. (74a) at $y = 0$ (or b) and using Eq. (68b) results in

$$\Phi_{,y} = 0 \qquad \text{at} \quad y = 0, b \tag{76b}$$

Also evaluating Eq. (65c) once at $x = 0$ (or a) and then at $y = 0$ (or b) and using Eqs. (68a) and (68b), respectively, results in

$$w_{,xx} = 0 \qquad \text{at} \quad x = 0, a \tag{77a}$$

$$w_{,yy} = 0 \qquad \text{at} \quad y = 0, b \tag{77b}$$

The traction-free boundary conditions, see Eqs. (69), can also be expressed in terms of Φ and w as follows: differentiate Eqs. (69a) and (69b) with respect to y and x, respectively, subtract the results, and substitute Eq. (71) in the final result to obtain

$$\Phi_{,z} = 0 \qquad \text{at} \quad z = \pm h/2 \tag{78}$$

Now differentiate Eqs. (69a) and (69b) with respect to x and y, add the results, and substitute the final result into Eq. (65c) to get

$$C_{33}w_{,zz} - C_{13}\nabla^2 w - \rho\ddot{w} = 0 \qquad \text{at} \quad z = \pm h/2 \tag{79}$$

The last conditions at $z = \pm h/2$ are obtained by substituting Eqs. (69c) into Eq. (75)

$$H_1\nabla^2 w_{,z} + H_2\ddot{w}_{,z} + H_3 w_{,zzz} = 0 \qquad \text{at} \quad z = \pm h/2 \tag{80}$$

where

$$H_1 = (C_{13}+C_{44})(C_{11}C_{33} - C_{13}^2) - C_{44}C_{13}^2$$
$$H_2 = -\rho[C_{44}(C_{13}+C_{33}) + C_{13}C_{33}] \tag{81}$$
$$H_3 = C_{13}C_{33}C_{44}$$

Hence, in summary, Eq. (70) is supplemented by the following conditions:

$$\Phi_{,x} = 0 \qquad \text{at} \quad x = 0, a \tag{82a}$$

$$\Phi_{,y} = 0 \qquad \text{at} \quad y = 0, b \tag{82b}$$

$$\Phi_{,z} = 0 \qquad \text{at} \quad z = h/2, -h/2 \tag{82c}$$

Equation (73), on the other hand, is accompanied by the conditions

$$w = w_{,xx} = 0 \qquad \text{at} \quad x = 0, a \tag{83a}$$

$$w = w_{,yy} = 0 \qquad \text{at} \quad y = 0, b \tag{83b}$$

$$C_{33}w_{,zz} - C_{13}\nabla^2 w - \rho\ddot{w} = 0 \qquad \text{at} \quad z = \pm h/2$$
$$H_1\nabla^2 w_{,z} + H_2\ddot{w}_{,z} + H_3 w_{,zzz} = 0 \qquad \text{at} \quad z = \pm h/2 \tag{83c}$$

Natural Frequencies According to Elasticity Equations

The preceding results reveal that the differential eigenvalue problem of the plate according to an exact theory is resolved into two distinct problems. The vibrational modes of the first problem, i.e., Eqs. (70) and (82), correspond to $w(x, y, z, t) = 0$ and those of the second problem, Eqs. (73) and (83), correspond to $\Phi(x, y, z, t) = 0$. We refer to these modes as the zero-w modes and zero-Φ modes, respectively. It is readily seen that the boundary conditions in Eqs. (82a) and (82b) are identically satisfied by the solution

$$\Phi(x, y, z, t) = \Phi_{mn}(z)\cdot\cos\alpha_m x \cos\beta_n y \cdot \cos\omega_{mn}t \tag{84}$$

Furthermore, substitution of Eq. (84) into Eq. (70) results in

$$C_{44}\frac{\mathrm{d}^2\Phi_{mn}}{\mathrm{d}z^2} - \left[\frac{1}{2}(C_{11}-C_{12})(\alpha_m^2+\beta_n^2) - \rho\omega_{mn}^2\right]\Phi_{mn} = 0 \tag{85}$$

The solution of Eq. (85), subject to the boundary conditions (82c), can be either symmetric or antisymmetric with respect to the middle surface of the plate. Therefore, there exist two different kinds of zero-w modes. In the first kind, the in-plane displacement components are symmetric with respect to the z-coordinate, whereas in the second kind, these components are antisymmetric. The frequencies obtained are

$$\frac{\rho}{G}\omega_{mn}^2 = \frac{Gz}{G}\left(\frac{j\pi}{h}\right)^2 + (\alpha_m^2 + \beta_n^2) \tag{86}$$

where $j = 0, 2, 4, \ldots$, for the first kind of zero-w modes and $j = 1, 3, \ldots$, for the second kind of zero-w modes. It is to be noted that the first kind of zero-w modes are, indeed, the stretching modes of the plate. The thickness-twist frequencies of the plate, on the other hand, correspond to the second kind of zero-w modes.

By comparing Eq. (61) with Eq. (86) we conclude that the first stretching frequency of the plate predicted by all of the equivalent single-layer plate theories is identical to the one obtained from the exact theory ($j = 0$). Also by comparing the first thickness-twist frequency of the exact theory ($j = 1$) with the ones obtained within TSDPT and FSDPT, we conclude that 1) the result of TSDPT is very accurate and 2) if $\pi^2/12$ is used for the shear correction factor, the result of FSDPT becomes an exact one. This latter conclusion was also reached by Srinivas et al.[12] for vibrations of a homogeneous isotropic plate.

Substitution of

$$w(x, y, z, t) = W_{mn}(z)\cdot\sin\alpha_m x \sin\beta_n y \cdot\cos\omega_{mn}t \tag{87}$$

which identically satisfies the boundary conditions (75a) and (83b), into Eq. (73) results in

$$a_0\frac{\mathrm{d}^4 W_{mn}}{\mathrm{d}z^4} + a_2\frac{\mathrm{d}^2 W_{mn}}{\mathrm{d}z^2} + a_4 W_{mn} = 0 \tag{88}$$

where

$$a_0 = C_{33}C_{44}$$
$$a_2 = (C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33})(\alpha_m^2+\beta_n^2) + \rho(C_{33}+C_{44})\omega_{mn}^2 \tag{89}$$
$$a_4 = [\rho\omega_{mn}^2 - C_{11}(\alpha_m^2+\beta_n^2)][\rho\omega_{mn}^2 - C_{44}(\alpha_m^2+\beta_n^2)]$$

Solution of Eq. (88), subject to the boundary conditions

$$C_{33}\frac{\mathrm{d}^2 W_{mn}}{\mathrm{d}z^2} + [C_{13}(\alpha_m^2+\beta_n^2) + \rho\omega_{mn}^2]W_{mn} = 0$$
$$\text{at } z = \pm h/2 \tag{90}$$
$$H_3\frac{\mathrm{d}^3 W_{mn}}{\mathrm{d}z^3} - [H_2\omega_{mn}^2 + H_1(\alpha_n^2+\beta_n^2)]\frac{\mathrm{d}W_{mn}}{\mathrm{d}z} = 0$$

also is either symmetric or antisymmetric with respect to thickness coordinate. Therefore, there are two different kinds of zero-Φ modes. In the first kind of zero-Φ modes the in-plane displacement components are antisymmetric and the transverse displacement is symmetric with respect to the z-coordinate. The corresponding frequencies are obtained by solving the transendental frequency equation

$$S_1K_2 \cosh(\gamma_1 h/2) \sinh(\gamma_2 h/2) - S_2K_1 \cosh(\gamma_2 h/2) \sin(h\gamma_1 h/2) = 0 \tag{91}$$

where

$$\begin{aligned} \gamma_1 &= \{(1/2a_0)[-a_2 + (a_2^2 - 4a_0a_4)^{1/2}]\}^{1/2} \\ \gamma_2 &= \{(1/2a_0)[-a_2 - (a_2^2 - 4a_0a_4)^{1/2}]\}^{1/2} \\ S_i &= C_{33}\gamma_i^2 + C_{13}(\alpha_m^2 + \beta_n^2) + \rho\omega_{mn}^2 \\ K_i &= H_3\gamma_i^3 - [H_2\omega_{mn}^2 + H_1(\alpha_m^2 + \beta_n^2)]\gamma_i, \qquad i = 1, 2 \end{aligned} \tag{92}$$

The flexural and flexural thickness shear frequencies obtained within TSDPT and FSDPT, from Eqs. (55) and (57), are approximations to the lowest frequencies determined from Eqs. (91).

In the zero-Φ modes of the second kind the in-plane displacement components are symmetric and the transverse displacement is antisymmetric with respect to the middle surface of the plate. The frequencies of such modes are obtained from the equation

$$S_1K_2 \sinh(\gamma_1 h/2) \cosh(\gamma_2 h/2) - S_2K_1 \sinh(\gamma_2 h/2) \cosh(\gamma_1 h/2) = 0 \tag{93}$$

The second kind of zero-Φ modes of the plate cannot be predicted by any of the equivalent single-layer plate theories. This, of course, is due to the assumption of transverse inextensibility made in these theories which forces the transverse displacement to be an even function of z. Surprisingly, however, the second stretching frequency of the plate according to the equivalent single-layer theories [see Eq. (62)] is found to be the approximate value of the lowest exact frequency obtained from Eq. (93). Remember, however, that the lowest flexural frequency of the plate corresponds to the zero-Φ modes of the first kind. In passing, it should be pointed out that in general the parameters γ_1 and γ_2 are complex valued and a procedure similar to the one introduced in Ref. 15 should be used for finding the zeros of the frequency equations (91) and (93).

Numerical Results and Discussions

In what follows we consider several numerical examples in which the natural frequencies of homogeneous and laminated plates obtained within the plate theories are compared with those determined from the exact theory of elasticity. The elasticity solution is checked by working out the example problems of a homogeneous isotropic plate and a three-ply laminated plate considered by Srinivas et al.[12]

Example 1

Consider a homogeneous plate made of transversely isotropic material whose properties are $E_1 = E_2\ (\equiv E) = 2E_3\ (\equiv 2E_z)$, $G_{13} = G_{23}\ (\equiv G_z) = 0.3\,E_3$; $\nu_{13} = \nu_{23}\ (\equiv \nu_z) = 0.4$, $\nu_{12} = \nu_{21}\ (\equiv \nu) = 0.25$; and $G_{12}\ (\equiv G) = [E/2(1+\nu)] = 0.8\ E_z$. It is to be noted that the x-y plane is the plane of isotropy. In Table 1 are tabulated the nondimensionalized natural frequencies of the plate as predicted by various theories. As we noted earlier, for each pair of the integers m and n the elasticity theory predicts an infinite number of frequencies. On the other hand, LWPT yields the lowest $3(N+1)$ frequencies of the plate for each (m, n). In Table 1 we have assumed that the homogeneous plate is composed of 10 identical mathematical layers. Therefore, in this example, LWPT will yield 33 lowest frequencies of the plate of which only the first 10 values are presented in Table 1. It is noted that all vibrational modes can be predicted by LWPT. In Table 1 we used the following abbreviations: zΦf for zero-Φ modes of first kind; zΦs for zero-Φ modes of second kind; zwf for zero-w modes of first kind; and zws for zero-w modes of second kind. Also

$$\frac{g^2}{\pi^2} = \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]h^2 \tag{94}$$

Table 1 Nondimensionalized natural frequencies of a homogeneous transversely isotropic plate (example 1): $\bar{\omega}_{mnk} = \omega_{mnk}h\sqrt{\rho/G}$

Theory	g^2/π^2	zΦf	zwf	zΦs	zws	zΦf	zΦs	zwf	zws	zΦf	zΦs
Exact	0.04	0.08673	0.44429	0.72376	1.9745	2.0636	3.7615	3.8732	4.1191	5.7885	5.7989
	0.1	0.19860	0.70248	1.1398	2.0481	2.2518	3.7211	3.9113	4.2687	5.8141	5.8395
	0.2	0.35368	0.99346	1.5997	2.1652	2.5267	3.7051	3.9738	4.4672	5.8564	5.9056
	0.16	0.29535	0.88858	1.4354	2.1191	2.4215	3.7069	3.9489	4.3922	5.8395	5.8794
	0.26	0.43404	1.1327	1.8142	2.2325	2.6749	3.7103	4.0109	4.5725	5.8816	5.9444
LWPT (N = 10)	0.04	0.08683	0.44429	0.72377	1.9822	2.0714	3.8046	3.9364	4.1560	6.0032	6.0140
	0.1	0.19892	0.70248	1.1399	2.0555	2.2597	3.7605	3.9738	4.3094	6.0278	6.0543
	0.2	0.35441	0.99346	1.5998	2.1722	2.5348	3.7420	4.0354	4.5108	6.0686	6.1199
	0.16	0.29591	0.88858	1.4355	2.1263	2.4295	3.7446	4.0109	4.4348	6.0523	6.0939
	0.26	0.43504	1.1327	1.8144	2.2394	2.6833	3.7463	4.0719	4.6173	6.0929	6.1584
TSDPT	0.04	0.08660	0.44429	0.72552	1.9757	2.0700					
	0.1	0.19801	0.70248	1.1471	2.0492	2.2658					
	0.2	0.35213	0.99346	1.6223	2.1663	2.5529					
	0.16	0.29418	0.88858	1.4510	2.1202	2.4428					
	0.26	0.43194	1.1327	1.8497	2.2336	2.7085					
FSDPT ($K^2 = \pi^2/12$)	0.04	0.08652	0.44429	0.72552	1.9745	2.0690					
	0.1	0.19759	0.70248	1.1471	2.0481	2.2650					
	0.2	0.35069	0.99346	1.6223	2.1652	2.5523					
	0.16	0.29321	0.88858	1.4510	2.1191	2.4422					
	0.26	0.42968	1.1327	1.8497	2.2325	2.7081					
CLPT	0.04	0.09305	0.44429	0.72552							
	0.1	0.23263	0.70248	1.1471							
	0.2	0.46526	0.99346	1.6223							
	0.16	0.37221	0.88858	1.4510							
	0.26	0.60483	1.1327	1.8497							

Table 2 Dimensionless natural frequencies of (0/90), (0/90/0), and (0/90/0/90) laminated square plates according to elasticity theory and LWPT: $\bar{\omega}_{mnk} = \omega_{mnk} h\sqrt{\rho/E_2}$, $a/h = 10$

	(0/90)			(0/90/0)				(0/90/0/90)		
Theory	$m=n=1$	$m=1, n=2$, $m=2, n=1$	$m=n=2$	$m=n=1$	$m=1, n=2$	$m=2, n=1$	$m=n=2$	$m=n=1$	$m=1, n=2$, $m=2, n=1$	$m=n=2$
Exact	0.06027	0.14539	0.20229	0.06715	0.12811	0.17217	0.20798	0.06621	0.15194	0.20841
	0.52994	0.62352	0.95796	0.50350	0.68880	0.58366	0.97517	0.54596	0.63875	1.0623
	0.58275	0.95652	1.0300	0.63775	0.95017	1.1780	1.2034	0.59996	1.0761	1.1557
	1.2367	1.2389	1.2396	1.2429	1.2406	1.2752	1.2916	1.2425	1.2417	1.2443
	1.2793	1.3189	1.6670	1.2790	1.3455	1.3141	1.4326	1.2988	1.3425	1.6227
	1.2977	1.6583	1.6997	1.3292	1.4189	1.7778	1.8037	1.3265	1.6323	1.6894
	2.4730	2.4676	2.4618	2.1533	2.3487	2.1724	2.3653	2.3631	2.3869	2.4769
	2.5658	2.5843	2.7243	2.4894	2.4839	2.4925	2.4877	2.3789	2.4844	2.5587
	2.5736	2.7110	2.7258	2.7419	2.7573	2.8899	2.9047	2.4911	2.5614	2.5942
	3.5811	3.5966	3.6301	3.5416	3.6251	3.5533	3.6361	3.6661	3.6778	3.6891
LWPT	0.06030	0.14554	0.20254	0.06716	0.12816	0.17225	0.20808	0.06622	0.15198	0.20848
	0.53009	0.62367	0.95878	0.50355	0.68887	0.58372	0.97560	0.54600	0.63879	1.0626
	0.58291	0.95744	1.0309	0.63782	0.95059	1.1790	1.2047	0.59999	1.0765	1.1561
	1.2404	1.2428	1.2436	1.2445	1.2423	1.2765	1.2927	1.2435	1.2426	1.2453
	1.2825	1.3220	1.6697	1.2812	1.3465	1.3164	1.4348	1.2996	1.3433	1.6238
	1.3010	1.6614	1.7031	1.3301	1.4210	1.7789	1.8048	1.3274	1.6336	1.6907
	2.5040	2.4988	2.4932	2.1621	2.3575	2.1811	2.3740	2.3698	2.3936	2.4843
	2.6039	2.6213	2.7615	2.5022	2.4970	2.5054	2.5008	2.3856	2.4917	2.5651
	2.6091	2.7487	2.7640	2.7612	2.7765	2.9089	2.9237	2.4983	2.5678	2.6007
	3.6743	3.6901	3.7298	3.5917	3.6756	3.6032	3.6864	3.6939	3.7051	3.7152

Table 3 Dimensionless natural frequencies of (0/90), (0/90/0), and (0/90/0/90) laminated square plates according to TSDPT, FSDPT, and centerline CLPT: $\bar{\omega}_{mnk} = \omega_{mnk} h\sqrt{\rho/E_2}$, $a/h = 10$

	(0/90)			(0/90/0)				(0/90/0/90)		
Theory	$m=n=1$	$m=1, n=2$, $m=2, n=1$	$m=n=2$	$m=n=1$	$m=1, n=2$	$m=2, n=1$	$m=n=2$	$m=n=1$	$m=1, n=2$, $m=2, n=1$	$m=n=2$
TSDPT	0.06057	0.14681	0.20482	0.06839	0.13010	0.17921	0.21526	0.06789	0.16065	0.22108
	0.53494	0.62837	0.99135	0.50897	0.69350	0.5886	1.0179	0.54845	0.64119	1.0822
	0.58827	0.99618	1.0738	0.64174	0.99373	1.2408	1.2835	0.60261	1.0993	1.1840
	1.4297	1.4716	1.8026	1.4106	1.4768	1.4429	1.5695	1.4237	1.4651	1.7323
	1.4588	1.8134	1.8683	1.4638	1.5598	1.9044	1.9300	1.4535	1.7525	1.8153
FSDPT	0.06038	0.14545	0.20271	0.06931	0.12886	0.18674	0.22055	0.06791	0.16066	0.22097
$K_4^2 = K_5^2$	0.53614	0.62949	0.99914	0.50897	0.69350	0.58856	1.0179	0.54750	0.64031	1.0755
$= \pi^2/12$	0.58943	1.0045	1.0821	0.64174	0.99373	1.2408	1.2835	0.60169	1.0919	1.1762
	1.4284	1.4704	1.7971	1.2719	1.4057	1.3039	1.4343	1.4233	1.4648	1.7369
	1.4577	1.8084	1.8639	1.5845	1.6134	2.0082	2.0309	1.4532	1.7565	1.8188
CLPT	0.06513	0.17744	0.25814	0.07769	0.15185	0.26599	0.31077	0.07474	0.20737	0.29824
	0.55291	0.64527	1.1161	0.50897	0.69350	0.58856	1.0179	0.55170	0.64426	1.1060
	0.60577	1.1297	1.2115	0.64174	0.99373	1.2408	1.2835	0.60577	1.1252	1.2115

The results of LWPT can approach those of the exact theory by increasing the number of mathematical layers replacing each physical ply in the laminate. This will be observed later from the numerical results of Table 4.

Remember that the first frequency corresponding to zΦs mode in Table 1 predicted by any of the equivalent single-layer theories is obtained from the stretching equations of the plate, rather than from the bending equations. As we pointed out earlier, the zΦs modes cannot be predicted by such equivalent single-layer theories.

Example 2

Here we consider symmetric and antisymmetric cross-ply laminates (see Jones[18]). Each ply is assumed to be orthotropic with nine independent material properties: $E_1 = 25.1 \times 10^6$ psi, $E_2 = 4.8 \times 10^6$ psi, and $E_3 = 0.75 \times 10^6$ psi; $G_{12} = 1.36 \times 10^6$ psi, $G_{13} = 1.2 \times 10^6$ psi, and $G_{23} = 0.47 \times 10^6$ psi; and $\nu_{12} = 0.036$, $\nu_{13} = 0.25$, and $\nu_{23} = 0.171$. In Tables 2 and 3 we have tabulated the first ten frequency parameters of (0/90), (0/90/0), and (0/90/0/90) laminates corresponding to each pair of integers m and n.

Each physical ply is modeled as being made up of six layers within LWPT. Hence, it is to be noted that, for example, for the (0/90/0/90) laminate LWPT yields the 75 lowest frequencies for each pair of m and n. In other words, in such a laminate we allowed for 75 generalized displacement functions; $u^i(x, y, t)$, $v^i(x, y, t)$, and $w^i(x, y, t)$, $i = 1, 25$ [see Eqs. (3)]. Since it is at our disposal to increase the number of degrees of freedom in LWPT, such a theory should not be compared with TSDPT and FSDPT in which the transverse shear deformation is also taken into account but only five degrees of freedom are allowed. That is, LWPT, although being mathematically a two-dimensional theory, should only be compared with the exact theory and the individual-layer plate theories.[8]

Example 3

Here we compare the results of LWPT for a (0/90) laminate with those of the exact theory and an individual-layer plate theory (ILPT), which is developed by Cho et al.[8] The on-axis material properties of each layer are: $E_1 = 40E_2 = 40E_3$, $G_{12} = G_{13} = 0.6\, E_2$, $G_{23} = 0.5\, E_2$, and $\nu_{12} = \nu_{13} = 0.25$. It is to be noted that such a material is a transversely isotropic mate-

Table 4 Fundamental natural frequencies of (0/90) laminated square plates according to exact theory, layerwise plate theory LWPT, and individual-layer plate theory (ILPT): $\bar{\omega} = \omega a^2 \sqrt{\rho/E_2}$

a/h	Exact	ILPT	LWPT $N = 12$	LWPT $N = 20$	LWPT $N = 30$
2	4.935	4.810	4.957	4.944	4.939
5	8.518	8.388	4.541	8.526	8.521
10	10.333	10.270	10.344	10.337	10.335
20	11.036	11.016	11.039	11.037	10.036
25	11.131	11.118	11.134	11.132	11.132
50	11.263	11.260	11.264	11.264	11.263
100	11.297	11.296	11.298	11.297	11.297

rial, with the y-z plane being the plane of isotropy. As we mentioned earlier, in the individual-layer plate theories, a set of equations of motion are derived for each individual layer of the laminate by making certain assumptions concerning the variation of either stress components or displacement components with respect to the thickness coordinate. Similar to the exact analysis, the continuity of the displacement and transverse stress components at the interfaces and the boundary conditions at the bounding planes of the laminate are enforced as part of the problem. This makes the individual-layer theories, although two dimensional, computationally expensive.

In the individual-layer plate theory (ILPT) of Cho et al.,[8] a third-order displacement field is assumed for any typical individual layer and for each additional layer in a laminate 11 additional degrees of freedom are added in the problem. In ILPT each homogeneous layer can also be treated as a multilayer plate whose layers will, of course, have identical material properties.

The fundamental natural frequencies of (0/90) laminates with various length/thickness values are presented in Table 4. The results of the exact theory and LWPT are compared with those obtained within ILPT by Cho et al.[8] The convergence of the LWPT's results is clearly put into evidence as each actual layer is replaced by an increasing number of mathematical layers. In Ref. 8 it is not indicated how each actual layer is modeled for generating the numbers in Table 4. Therefore, no conclusion can be drawn here concerning the accuracy of ILPT as compared to that of LWPT. This comparison could have been made simply based on the total degrees of freedom allowed in each theory. However, we are inclined to prefer LWPT over ILPT merely because LWPT is computationally less time consuming.

Finally, it is to be noted that the results of LWPT and ILPT are upper and lower bounds of the exact solutions, respectively.

Concluding Remarks

In this study, Reddy's layerwise theory is used to perform free-vibration analysis of laminated plates. This theory is the most current and sophisticated theory in which full account is given to various three-dimensional effects. The results obtained from this theory are compared with those obtained from a full fledged three-dimensional elasticity analysis and various equivalent single-layer theories that are available. These include the classical laminated plate theory (CLPT), the first-order shear deformation laminated plate theory (FSDPT), and the third-order shear deformation plate theory (THSDPT). The elasticity equations are solved by utilizing the state-space variables and the transfer matrix. A detailed analysis is carried out, by uncoupling the Navier equations, to study the various mode shapes and natural frequencies of a homogeneous transversely isotropic plate. Results are also obtained for symmetric and antisymmetric cross-ply laminates.

For all of the cases studied, the layerwise theory yields very accurate results as compared to those obtained by solving the elasticity equations.

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